2. Structural Properties of Utility Representations

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Overview

1. What’s the point?

2. Continuity

3. Convexity

4. Monotonicity and Insatiability

5. Homotheticity

6. Separability

7. Quasilinearity

8. More

Overview

What's the point?

Properties of *u*

- *u* carries several implications for behaviour (warranted or not)
- Undertanding implications often allows *testing* model through its *identifying* assumptions
- **Models as maps**, simplified description of reality
  - Behavioural implications = Empirical content

Notation

Throughout: (X, d) is metric space

Open ε-neighbourhood of x in X: $B_\varepsilon(x) := \{y \in X \mid d(x, y) < \varepsilon\}$

Overview

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4. Monotonicity and Insatiability
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Continuity

Why continuity? Technical property, guarantees $\arg\max_{x \in A} u(x) \neq$ for A compact

$\implies$ Choices of decision-maker (DM) well-defined

Weierstrass Extreme Value Theorem

Let (X, d_X) and (Y, d_Y) be two metric spaces. If $f : X \to Y$ is a continuous function and S a compact set in (X, d_X), then f attains a maximum and a minimum in S: $\arg\max_{x \in S} f(x) \neq \emptyset$ and $\arg\min_{x \in S} f(x) \neq \emptyset$.

Continuity of Preferences

Definition

A preference relation $\succsim$ on X is **continuous** if for any two converging sequences, $\{x_n\}_n, \{y_n\}_n \subseteq X$, $x_n \to x$ and $y_n \to y$, such that $x_n \succsim y_n \; \forall n$, we have $x \succsim y$.

Lemma

Let $\succsim$ be a preference relation on X, and $\succ$ its asymmetric part. The following statements are equivalent:

(i) $\succsim$ is continuous;

(ii) $\forall x \in X$, $X_{x\succsim}$ and $X_{\succsim x}$ are closed;

(iii) $\forall x \in X$, $X_{x\succ}$ and $X_{\succ x}$ are open;

(iv) $\forall x, y \in X : x \succ y$, $\exists \varepsilon > 0$ s.t. $\forall x' \in B_\varepsilon(x), y' \in B_\varepsilon(y), x' \succ y'$.

(Left as an exercise.)

Continuous Utility Representation

Debreu's Theorem (1954, 1964)

Let $\succsim$ be a preference relation on X, and suppose X admits a countable, $\succsim$-dense subset Z. Then, $\succsim$ is continuous $\iff$ $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$.

Debreu's theorem is one of the most fundamental results in economics.

The theorem requires only X be a topological space that is (i) separable (admitting a countable, dense subset) and (ii) connected (not represented by the union of disjoint nonempty sets).

We'll prove the following (easier) version:

Debreu's Theorem (1954, 1964)

Let (X, d) be a convex metric space s.t. $\forall \alpha \in [0, 1]$, $d(\alpha x + (1 - \alpha)y, y) \geq \alpha d(x, y)$. Let $\succsim$ be a preference relation on X, and suppose X admits a countable, $\succsim$-dense subset Z. Then, $\succsim$ is continuous $\iff$ $\succsim$ admits a continuous utility representation $u : X \to \mathbb{R}$.

Continuous Utility Representation

Proof

$\Longleftarrow$: (if)

Take any $\{x_n\}_n, \{y_n\}_n \subseteq X$ s.t. $x_n \to x, y_n \to y$, and $x_n \succsim y_n$.

Then, $u(x_n) - u(y_n) \geq \mathbf{0}$, $\forall n$.

By continuity of u, $\lim_{n\to\infty} u(x_n) - u(y_n) = u(x) - u(y) \geq \mathbf{0} \implies x \succsim y$.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if)

Assume $\exists x, y \in X : x \succ y$ (ow just set $u(x) = c$).

We will prove this part in three steps:

1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.
2. Construct a utility function $u : X \to \mathbb{R}$ s.t. $u(Z)$ is dense in $[\mathbf{0}, \mathbf{1}]$.
3. Show u is continuous.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) **1. Show** $\forall x, y \in X : x \succ y, \exists z \in Z : x \succ z \succ y$**.**

(i) WTS $\exists\, x' \in X : x \succ x' \succ y$

- For $a \in [0,1]$, let $x_a := ax + (1-a)y \in X$ (by convexity of X).
- Define $A := \{a \in [0,1] \mid x_a \succsim x\}$ and $\alpha := \inf A$.

  A nonempty and bounded below: $1 \in A$ ($\because \succsim$ complete); $0 \notin A$ ($\because x \succ y$)
- WTS $x_\alpha \sim x$. Suppose not.

$$\begin{aligned} \text{If } x_\alpha \succ x &\implies \exists \varepsilon > 0 : x_{\alpha-\varepsilon} \succ x \qquad \text{by continuity Lemma} \\ &\implies \alpha - \varepsilon \in A \\ &\implies \alpha \neq \inf A, \end{aligned}$$

a contradiction. If instead,

$$\begin{aligned} x \succ x_\alpha &\implies \exists \varepsilon' > 0 : x \succ x_{\alpha+\varepsilon},\ \forall 0 < \varepsilon \leq \varepsilon' \quad \text{by continuity Lemma} \\ &\implies \alpha + \varepsilon > \inf A \\ &\implies \alpha \neq \inf A, \end{aligned}$$

again a contradiction. $\implies x \sim x_\alpha$ (by completeness).

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) **1. Show** $\forall x, y \in X : x \succ y, \exists z \in Z : x \succ z \succ y$**.**

(i) WTS $\exists\, x' \in X : x \succ x' \succ y$

- For $a \in [0,1]$, let $x_a := ax + (1-a)y \in X$.
- Define $A := \{a \in [0,1] \mid x_a \succsim x\}$ and $\alpha := \inf A$.
- $x_\alpha \sim x$.
- By definition:

$$\begin{aligned} x \sim x_\alpha \succ y &\implies \alpha > 0 \\ &\implies \forall \alpha' \in (0, \alpha), \alpha' \notin A \quad (\because \alpha = \inf A) \\ &\implies x \succ x_{\alpha'} \quad (\because \text{ completeness}) \end{aligned}$$

- By continuity Lemma, $\exists \varepsilon' > 0$ s.t. $\forall x' \in B_{\varepsilon'}(x_\alpha)$
  $\implies \exists \lambda \in (0,1)\; d(\lambda x_\alpha + (1-\lambda)y, x_\alpha) \leq (1-\lambda) d(x_\alpha, y) < \varepsilon'$.
- Define $\alpha' := \lambda\alpha \in (0, \alpha)$.
  $\alpha' \notin A$ and $x_{\alpha'} \in B_{\varepsilon'}(x_\alpha) \implies x \succ x_{\alpha'} \succ y$

Continuous Utility Representation

Proof

$\implies$: (only if) **1. Show** $\forall x, y \in X : x \succ y, \exists z \in Z : x \succ z \succ y$**.**

(i) WTS $\exists\, x' \in X : x \succ x' \succ y$

(ii) Find $z \in Z : x \succ z \succ y$.

- Z is $\succsim$-dense in X: $x_{\alpha'} \succ y \implies \exists z \in Z : x_{\alpha'} \succsim z \succ y$

  $\implies \exists z \in Z : x \succ x_{\alpha'} \succsim z \succ y$

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) 1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.

2. Construct a utility function $u : X \to \mathbb{R}$ s.t. $u(Z)$ is dense in $[0,1]$.

Remove maximal and minimal elements of X from Z: $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$.

(i) Fix order on $Z = \{z_1, z_2, ...\}$ and let $Z_n := \{z_1, ..., z_{n-1}\}$ for $n \geq 2$. Define u on Z recursively.

$u(z_1) := 1/2$. For $n > 1$,

(a) if $\exists z_m \in Z_n$ s.t. $z_n \sim z_m$, set $u(z_n) := u(z_m)$;

(b) if $z_n \succ z_m \ \forall z_m \in Z_n$, then set $u(z_n) := (1 + \max_{z \in Z_n} u(z))/2$;

(c) if $z_m \succ z_n \ \forall z_m \in Z_n$, then set $u(z_n) := (0 + \min_{z \in Z_n} u(z))/2$;

(d) if neither (a)-(c) hold, then $\exists z_\ell, z_m \in Z_n$ s.t. $z_\ell \succ z_n \succ z_m$ and $\nexists z' \in Z_n : z_\ell \succ z \succ z_n$ nor $z_n \succ z \succ z_m$,
and in such case set $u(z_n) := (\min_{z \in Z_n : z \succ z_n} u(z) + \max_{z \in Z_n : z_n \succ z} u(z))/2$.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) 1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.
2. Construct a utility function $u : X \to \mathbb{R}$ **s.t.** $u(Z)$ **is dense in** $[0, 1]$**.**

Remove maximal and minimal elements of X from Z: $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$.

(i) Fix order on $Z = \{z_1, z_2, ...\}$ and let $Z_n := \{z_1, ..., z_{n-1}\}$ for $n \geq 2$. Define u on Z recursively.

(ii) WTS $u(Z)$ is dense in $[0, 1]$.

- By 1., $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.

$\Longrightarrow$ $\forall z_n, z_m \in Z$ s.t. $z_n \succ z_m$, there is $\ell, \ell', \ell'' > n, m$ such that $z_\ell \succ z_n \succ z_{\ell'} \succ z_m \succ z_{\ell''}$
where z_ℓ and $z_{\ell''}$ exist because we removed the maximal and minimal elements of X from Z

- By construction, $u(Z)$ = set of dyadic numbers in $(0, 1) := \{m/2^n \mid m, n \in \mathbb{N},\ m < 2^n\}$, which is dense in $[0, 1]$.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) 1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.

2. Construct a utility function $u : X \to \mathbb{R}$ **s.t.** $u(Z)$ **is dense in** $[0, 1]$**.**

Remove maximal and minimal elements of X from Z: $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$.

(i) Fix order on $Z = \{z_1, z_2, ...\}$ and let $Z_n := \{z_1, ..., z_{n-1}\}$ for $n \geq 2$. Define u on Z recursively.

(ii) WTS $u(Z)$ is dense in $[0, 1]$.

(iii) WT extend u to X.

- $\forall x \in \arg\max_{\succsim} X$ and $\forall y \in \arg\min_{\succsim} X$ we can assign $u(x) := 1$ and $u(y) := 0$.
- Set $u(x) := \sup\{u(z) \mid z \in Z \text{ and } x \succ z\} = \sup_{z \in Z_{x\succ}} u(z)$.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) 1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.

2. Construct a utility function $u : X \to \mathbb{R}$ **s.t.** $u(Z)$ **is dense in** $[0, 1]$**.**

Remove maximal and minimal elements of X from Z: $Z \cap (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X) = \emptyset$.

(i) Fix order on $Z = \{z_1, z_2, ...\}$ and let $Z_n := \{z_1, ..., z_{n-1}\}$ for $n \geq 2$. Define u on Z recursively.

(ii) WTS $u(Z)$ is dense in $[0, 1]$.

(iii) WT extend u to X.

(iv) WTS u represents $\succsim$.

- By 1., $x \succ y \implies \exists z, z' \in Z$ s.t. $x \succ z \succ z' \succ y \implies u(x) \geq u(z) > u(z') \geq u(y)$.
- By definition, $x \sim y \implies u(x) = u(y)$.

Continuous Utility Representation

Proof

$\Longrightarrow$: (only if) 1. Show $\forall x, y \in X : x \succ y$, $\exists z \in Z : x \succ z \succ y$.

2. Construct a utility function $u : X \to \mathbb{R}$ s.t. $u(Z)$ is dense in $[0, 1]$.

3. Show u is continuous.

(i) Take any $x \in X \setminus (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X)$.

- By 1. and 2., for any $\varepsilon > 0$, $\exists z, z' \in Z : u(x) - \varepsilon < u(z) < u(x) < u(z') < u(x) + \varepsilon$.
- By continuity Lemma, $\exists \delta > 0 : \forall x' \in B_\delta(x)$, $u(x) - \varepsilon < u(z) < u(x') < u(z') < u(x) + \varepsilon$.

(ii) Take any $x \in (\arg\max_{\succsim} X \cup \arg\min_{\succsim} X)$.

- Suppose $x \in \arg\max_{\succsim} X$.
- By 1. and 2., for any $\varepsilon > 0$, $\exists z \in Z : 1 - \varepsilon = u(x) - \varepsilon < u(z) < u(x) = 1$.
- By continuity Lemma, $\exists \delta > 0 : \forall x' \in B_\delta(x)$, $u(x) - \varepsilon < u(x') \leq u(x) = 1$.
- Argument for $x \in \arg\min_{\succsim} X$ is symmetric.

□

Continuous Utility Representation

Existence of *a* continuous u-representation does **not** mean that any utility representation of $\succsim$ is continuous.

Example: $\succsim \subseteq [0, 2]^2 : x \succsim y \iff x \geq y$.

u = id represents $\succsim$, but so does discontinuous function

$v : x \mapsto 1_{\{x<1\}}\, x + 1_{\{x=1\}}\, 2 + 1_{\{x>1\}}\, 3x$

Overview

Convexity

Definition

A real-valued function u on a convex set X is **(strictly) quasiconcave** if $\forall x, y \in X$ and $\forall \lambda \in [0, 1]$ (resp. $\lambda \in (0, 1)$), $u(\lambda x + (1 - \lambda y)) \geq (>) \min\{u(x), u(y)\}$.

Definition

A preference relation $\succsim$ on a convex set X is **convex** iff for any $x \succsim y$ and any $\lambda \in [0, 1]$, we have that $\lambda x + (1 - \lambda)y \succsim y$.
It is **strictly convex** if, in addition, $\forall x \succsim y, x \neq y$, and any $\lambda \in (0, 1)$, $\lambda x + (1 - \lambda)y \succ y$.

Choose mixtures over extremes.

Convexity

Proposition

Let $\succsim$ be preference relation on convex set X and $u : X \to \mathbb{R}$ a utility representation. The following statements are equivalent:

(i) $\succsim$ is convex;

(ii) $X_{\succsim y}$ is convex $\forall y \in X$;

(iii) u is quasiconcave;

(iv) $\{x \in X \mid u(x) \geq \bar{u}\}$ is convex $\forall \bar{u} \in \mathbb{R}$.

Moreover, $\succsim$ is strictly convex if and only if u is strictly quasiconcave.

Convexity

Proof

(i) $\succsim$ is convex $\implies$ (ii) $X_{\succsim y}$ is convex $\forall y \in X$:
Take any $x, x' \in X_{\succsim y}$ and let, without loss of generality (by completeness), $x \succsim x'$.
Then $\lambda x + (1 - \lambda)x' \succsim x' \succsim y \; \forall \lambda \in [0, 1]$ (by convexity and transitivity).

(i) $\succsim$ is convex $\impliedby$ (ii) $X_{\succsim y}$ is convex $\forall y \in X$:
By completeness, $y \in X_{\succsim y}$.
$X_{\succsim y}$ is convex $\implies \forall x \in X_{\succsim y}$ and $\lambda \in [0, 1]$, $\lambda x + (1 - \lambda)y \succsim y$.

Convexity

Proof

(i) $\succsim$ is convex $\iff$ (iii) u is quasiconcave:
Take any $x, y \in X$ such that $x \succsim y \iff u(x) \geq u(y)$, and any $\lambda \in [0, 1]$.

$$\begin{aligned} \succsim \text{ convex} &\iff \lambda x + (1-\lambda)y \succsim y \\ &\iff u(\lambda x + (1-\lambda)y) \geq u(y) = \min\{u(x), u(y)\} \\ &\iff u \text{ quasiconcave.} \end{aligned}$$

For strict convexity of $\succsim$ and strict quasiconcavity of u, replace $\succsim$ and $\geq$ with $\succ$ and $>$.

Convexity

Proof

(iii) u is quasiconcave $\implies$ (iv) $\{x \in X \mid u(x) \geq \bar{u}\}$ is convex $\forall \bar{u} \in \mathbb{R}$:

$\forall x, y \in X : u(x), u(y) \geq \bar{u}$,

$u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\} \geq \bar{u}, \forall \lambda \in [0, 1]$ (by quasiconcavity of u).

(iii) u is quasiconcave $\impliedby$ (iv) $\{x \in X \mid u(x) \geq \bar{u}\}$ is convex $\forall \bar{u} \in \mathbb{R}$:

Take any $x, y \in X$.

$\{z \in X \mid u(z) \geq \min\{u(x), u(y)\}\}$ convex

$\implies \forall \lambda \in [0, 1], \lambda x + (1 - \lambda)y \in \{z \in X \mid u(z) \geq \min\{u(x), u(y)\}\}$;

$\implies u(\lambda x + (1 - \lambda)y) \geq \min\{u(x), u(y)\}$. □

Convexity of Preferences and Choice

Theorem

Let $\succsim$ be a convex preference relation on a convex set X.
Then, for any convex $A \in \mathbf{2}^X$, $\arg\max_{\succsim} A$ is convex.
If, in addition, $\succsim$ is strictly convex, then $\arg\max_{\succsim} A$ contains at most one element.

(Proof is left as an exercise.)

Overview

Monotonicity and Insatiability

Monotonicity: choose more over less (e.g., money)

Notation: for clarity, $x \gg y \iff x_i > y_i \ \forall i$

($>$ can be mistaken with the asymmetric part of $\geq$)

Definition

(i) $\succsim$ is **monotone** iff $x \geq y \implies x \succsim y$;

(ii) $\succsim$ is **strongly monotone** iff $x \geq (\gg) y \implies x \succsim (\succ) y$;

(iii) $\succsim$ is **strictly monotone** iff $x > y$ (i.e., $x \geq y$ and $x \neq y$) $\implies x \succ y$.

Let $\succsim$ be preference relation on $X \subseteq \mathbb{R}^k$ and $u : X \to \mathbb{R}$ a utility representation of $\succsim$.

(i) $\succsim$ is **monotone** if and only if $x \geq y \implies u(x) \geq u(y)$;

(ii) $\succsim$ is **strongly monotone** if and only if $x \geq (\gg) y \implies u(x) \geq (>) u(y)$;

(iii) $\succsim$ is **strictly monotone** if and only if $x > y$ ($x \geq y$ and $x \neq y$) $\implies u(x) > u(y)$.

Monotonicity and Insatiability

Definition

Let $\succsim$ be a preference relation on $X \subseteq \mathbb{R}^k$ and $u : X \to \mathbb{R}$ a utility representation of $\succsim$.

(i) $\succsim$ is **globally non-satiated** iff $\forall x \in X, \exists y \in X : y \succ x$.

(ii) $\succsim$ is **locally non-satiated** iff $\forall x \in X$ and $\varepsilon > 0$, $\exists y \in B_\varepsilon(x) : y \succ x$.

'Insatiability': improvability

- strict monotonicity $\implies$ strong monotonicity $\implies$ monotonicity
- strong monotonicity $\implies$ local non-satiation $\implies$ global non-satiation

Overview

Homotheticity

Definition

A preference relation $\succsim$ on $X = \mathbb{R}^k$ is **homothetic** iff $x \succsim y \implies \alpha x \succsim \alpha y, \forall \alpha \geq 0$.

Property that is quite important for aggregate demand to behave as if arising from choices of a representative consumer.

Proposition

Let $\succsim$ be a continuous, homothetic, and strongly monotone preference relation on $X = \mathbb{R}^k$. Then, it admits a continuous utility representation $u : X \to \mathbb{R}$ that is homogeneous of degree one.

(Proof is left as an exercise.)

Examples Examples of known utility functional forms that imply homothetic preferences?

How should we change the conditions of the propositon to get u as homogeneous of degree 2? And degree $k > 1$?

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Separability

Often we alternatives have different dimensions which are assessed separately.

'All else equal'...

- Choose job with better pay over job with less pay.
- Choose laptop with more memory over laptop with less memory.

  Not necessarily monotone:

- Choose phone with dimensions closer to my ideal dimensions.
  (Not larger phone is better, nor smaller phone is better.
  Maybe even multiple ideal points: want large phone or small phone, but not intermediate nor too large nor too small.)

Not always true...

- Choosing dessert with more chocolate does not mean 'all else equal' I choose more chocolate over less in all that I eat.

How to capture this? What implications does it have for utility representation? How to test this from choice data?

Weak Separability

Multi-dimensional alternatives: $X := \times_{i \in [n]} X_i \times \overline{X}$, where each X_i is a dimension and $[n] = \{1, ..., n\}$.

All else: Write $x_{-i} \in X_{-i} := \times_{j \in [n] \setminus \{i\}} X_j \times \overline{X}$ and $x = (x_i, x_{-i})$.

A preference relation on X is **weakly separable** in $\times_{i \in [n]} X_i$ iff, $\forall i \in [n]$, $\forall x_i, y_i \in X_i$ and $\forall x_{-i}, y_{-i} \in X_{-i}$, $\quad (x_i, x_{-i}) \succsim (y_i, x_{-i}) \iff (x_i, y_{-i}) \succsim (y_i, y_{-i})$.

(Does this capture what we wanted it to capture? What do you expect the utility representation to look like?)

Theorem

Let $\succsim$ be a preference relation on $X = \times_{i \in [n]} X_i \times \overline{X}$ admitting a utility representation $u : X \to \mathbb{R}$.

$\succsim$ is weakly separable in $\times_{i \in [n]} X_i$ if and only if $\exists v, \{u_i\}_{i \in [n]}$, such that

(i) $v : \times_{i \in [n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ and $u_i : X_i \to \mathbb{R}$ $\forall i$,

(ii) $u(x) = v(u_1(x_1), ..., u_n(x_n), \overline{x})$, and

(iii) v is strictly increasing in its first n arguments.

Weak Separability

Theorem

Let $\succsim$ be a preference relation on $X = \times_{i\in[n]} X_i \times \overline{X}$ admitting a utility representation $u : X \to \mathbb{R}$.

$\succsim$ is weakly separable in $\times_{i\in[n]} X_i$ if and only if $\exists v, \{u_i\}_{i\in[n]}$, such that

(i) $v : \times_{i\in[n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ and $u_i : X_i \to \mathbb{R}\ \forall i$,

(ii) $u(x) = v(u_1(x_1), ..., u_n(x_n), \overline{x})$, and

(iii) v is strictly increasing in its first n arguments.

Proof

$\Longleftarrow$: (if) Straightforward – left as an exercise.

$\Longrightarrow$: (only if) We break the proof into steps

Weak Separability

Proof

$\Longrightarrow$: (only if) We break the proof into steps:

(i) Define u_i: Fix $x^* \in X$. For $i \in [n]$, let $u_i(x_i) := u(x_i, x^*_{-i})$.

(ii) WTS: $\forall x, y \in X$ s.t. $\bar{x} = \bar{y}$, $\quad u_i(x_i) \geq u_i(y_i)\ \forall i \in [n] \implies u(x) \geq u(y)$.

$$u_i(x_i) \geq u_i(y_i)\ \forall i \in [n] \iff u(x_i, x^*_{-i}) \geq u(y_i, x^*_{-i})\ \forall i \in [n]$$

$$\iff (y_1, ..., y_{i-1}, x_i, x_{i+1}, ..., x_n, \bar{x}) \succsim (y_1, ..., y_{i-1}, y_i, x_{i+1}, ..., x_n, \bar{x})\ \forall i \in [n] \qquad \text{(weak sep.)}$$

i.e.,

$$(1)\quad (x_1, x_2, ..., x_n, \bar{x}) \succsim (y_1, x_2, ..., x_n, \bar{x})$$

$$(2)\quad (y_1, x_2, x_2, ..., x_n, \bar{x}) \succsim (y_1, y_2, x_2, ..., x_n, \bar{x})$$

$$\vdots$$

$$(n)\quad (y_1, y_2, ..., y_{n-1}, x_n, \bar{x}) \succsim (y_1, y_2, ..., y_{n-1}, y_n, \bar{x})$$

$$(1)-(n) \implies x = (x_1, x_2, ..., x_n, \bar{x}) \succsim (y_1, x_2, ..., x_n, \bar{x}) \succsim (y_1, y_2, ..., x_n, \bar{x}) \succsim \cdots$$

$$\succsim (y_1, y_2, ..., y_n, \bar{y}) = y$$

$$\implies x \succsim y \qquad \text{(by transitivity)}$$

$$\implies u(x) \geq u(y).$$

Weak Separability

Theorem

Let $\succsim$ be a preference relation on $X = \times_{i\in[n]} X_i \times \overline{X}$ admitting a utility representation $u : X \to \mathbb{R}$.

$\succsim$ is weakly separable in $\times_{i\in[n]} X_i$ if and only if $\exists v, \{u_i\}_{i\in[n]}$, such that

(i) $v : \times_{i\in[n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ and $u_i : X_i \to \mathbb{R}$ $\forall i$,

(ii) $u(x) = v(u_1(x_1), ..., u_n(x_n), \overline{x})$, and

(iii) v is strictly increasing in its first n arguments.

Proof

$\implies$: (only if) We break the proof into steps:

(i) Define u_i: Fix $x^* \in X$. For $i \in [n]$, let $u_i(x_i) := u(x_i, x^*_{-i})$.

(ii) WTS: $\forall x, y \in X$ s.t. $\overline{x} = \overline{y}$, $\quad u_i(x_i) \geq u_i(y_i)\ \forall i \in [n] \implies u(x) \geq u(y)$.
Moreover, if $\exists i : u_i(x_i) > u_i(y_i)$, then $u(x) > u(y)$.

(iii) Define v: For any $r \in \mathbb{R}^n : r_i \in u_i(X_i)\ \forall i \in [n]$, pick $x_i \in X_i$ s.t. $u_i(x_i) = r_i$.
For any $\overline{x} \in \overline{X}$, and for any $r \in \times_{i\in[n]} u_i(X_i)$, let $v(r, \overline{x}) := u(x)$.
By (ii), v is strictly increasing in r.

□

Separability

Theorem

Let $\succsim$ be a preference relation on $X = \times_{i\in[n]} X_i \times \overline{X}$ admitting a utility representation $u : X \to \mathbb{R}$.

$\succsim$ is weakly separable in $\times_{i\in[n]} X_i$ if and only if $\exists v, \{u_i\}_{i\in[n]}$, such that

(i) $v : \times_{i\in[n]} u_i(X_i) \times \overline{X} \to \mathbb{R}$ and $u_i : X_i \to \mathbb{R}\ \forall i$,

(ii) $u(x) = v(u_1(x_1), ..., u_n(x_n), \overline{x})$, and

(iii) v is strictly increasing in its first n arguments.

Examples of known utility functional forms that imply weakly separable preferences?

What is the role of $\overline{X}$?

Is additive utility ($u(x) = \sum_i u_i(x_i)$) weakly separable?

How should we change the conditions of the theorem to get additive utility?

Weak separability does not deliver additive separability...

Strong Separability

Definition

A preference relation $\succsim$ on $X = \times_{i\in[n]} X_i$ is **strongly separable** if $\forall I \subsetneq [n]$, $\forall x_I, y_I \in \times_{i\in I} X_I$ and $\forall x_{-I}, y_{-I} \in \times_{i\in[n]\setminus I} X_i =: X_{-I}$, we have that $(x_I, x_{-I}) \succsim (y_I, x_{-I}) \iff (x_I, y_{-I}) \succsim (y_I, y_{-I})$.

We now restrict to $X = \times_{i\in[n]} X_i$.

What else changed?

We'll also need the following:

Definition

$i \in [n]$ is an **essential component** if $\exists x_i, y_i \in X_i$ and $x_{-i} \in X_{-i}$ such that $(x_i, x_{-i}) \succ (y_i, x_{-i})$.

Theorem (Debreu 1960)

Let $\succsim$ be a preference relation on $X = \times_{i\in[n]} X_i$ admitting a utility representation $u : X \to \mathbb{R}$. Suppose there are at least three essential components.
$\succsim$ is strongly separable if and only if there are $\{u_i\}_{i\in[n]}$, where $u_i : X_i \to \mathbb{R}$, such that $u(x) = \sum_{i\in[n]} u_i(x_i)$.

Overview

Quasilinearity

Quasilinear utility: $u : Y \times \mathbb{R} \to \mathbb{R}$ s.t. $u(y, m) = v(y) + m$, with $v : Y \to \mathbb{R}$.

Interpretation: y as specific good, m money (available to acquire other goods).

Recurrently assumed, e.g., in contract theory, auctions, and mechanism design.

What are we assuming when we write down a quasilinear utility function?

Quasilinearity

Theorem

Let $\succsim$ be a preference relation on $Y \times \mathbb{R}$. $\succsim$ admits a quasilinear utility representation if and only if it satisfies the following properties:

(1) (money is good) $\forall y \in Y, m, m' \in \mathbb{R}$:
$$m' \geq m \iff (y, m') \succsim (y, m);$$

(2) (no wealth effects) $\forall y, y' \in Y, m, m', m'' \in \mathbb{R}$:
$$(y, m) \succsim (y', m') \iff (y, m + m'') \succsim (y', m' + m'');$$

(3) (money can compensate) $\forall y, y' \in Y, \exists m, m' \in \mathbb{R}$ s.t. $(y, m) \sim (y', m')$.

Proof

$\implies$ (only if):

(1) $m' \geq m \iff v(y) + m' \geq v(y) + m \iff (y, m') \succsim (y, m), \forall y \in Y, m, m' \in \mathbb{R}$ (£ good);

(2) $(y, m) \succsim (y', m') \iff v(y) + m \geq v(y') + m' \iff v(y) + m + m'' \geq v(y') + m' + m'' \iff (y, m + m'') \succsim (y', m' + m''), \forall y, y' \in Y, m, m', m'' \in \mathbb{R}$ (no £effect);

(3) $\forall y, y' \in Y, \exists m, m' \in \mathbb{R}$ such that $v(y) - v(y') = m' - m \iff v(y) + m = v(y') + m' \iff (y, m) \sim (y', m')$ (£ compensate).

Quasilinearity

Proof

$\Longleftarrow$ (if): Fix $(y^*, m^*) \in Y \times \mathbb{R}$.

Step 1: $\exists! \rho : Y \to \mathbb{R}$ s.t. $(y, \rho(y)) \sim (y^*, m^*)$.

- WTS existence

  By (3 / £ compensate), $\exists m(y), m'(y) \in \mathbb{R} : (y, m(y)) \sim (y^*, m'(y))$.

  By (2 / no wealth effects), $(y, m(y) - m'(y) + m^*) \sim (y^*, m^*)$.

  Define $\rho(y) := m(y) - m'(y) + m^*$.

- WTS uniqueness

  Suppose not unique: $\exists \nu \neq \rho : Y \to \mathbb{R}$ s.t. $(y, \nu(y)) \sim (y^*, m^*)\ \forall y \in Y$

  $\rho \neq \nu \implies \exists y' \in Y : \nu(y') \neq \rho(y')$.

  Suppose $\nu(y') > \rho(y')$. (argument for $\nu(y') < \rho(y')$ symmetric)

  (1 / £ good) implies:

  $\nu(y') > \rho(y') \implies (y', \nu(y')) \succ (y', \rho(y')) \sim (y^*, m^*) \sim (y', \nu(y'))$.

  $\implies (y', \nu(y')) \succ (y', \nu(y'))$, contradicting reflexivity of $\succsim$.

Quasilinearity

Proof

$\Longleftarrow$ (if): Fix $(y^*, m^*) \in Y \times \mathbb{R}$.

Step 1: $\exists! \rho : Y \to \mathbb{R}$ s.t. $(y, \rho(y)) \sim (y^*, m^*)$.

Step 2: Characterise v.

Define $v(y) := -\rho(y)$. WTS quasilinear function represents $\succsim$.

$$
\begin{aligned}
& (y,m) \succsim (y',m') \\
\iff & (y, m - m' + \rho(y')) \succsim (y', \rho(y')) \sim (y^*, m^*) && \text{Step 1 and (2 / no wealth effects)} \\
\iff & m - m' + \rho(y') \geq \rho(y) && \text{(1 / £ good)} \\
\iff & -\rho(y) + m \geq -\rho(y') + m' \\
\iff & v(y) + m \geq v(y') + m'.
\end{aligned}
$$

□

Overview

1. What's the point?
2. Continuity
3. Convexity
4. Monotonicity and Insatiability
5. Homotheticity
6. Separability
7. Quasilinearity
8. More

More

- Homotheticity and structural change: Comin, Lashkari, & Mestieri (2021 Ecta).
- Intertemporal choice and discounting: characterisation by Koopmans (1960 Ecta).
- Discounted past and future discounting, anticipated regret, time inconsistency: Ray, Vellodi, & Wang (2024 JEEA).
- Time $\neq$ risk preferences via quasilinear preferences: Alaoui & Penta (2024 WP).
- Refere-dependence: Masatlioglu & Ok (2005 JET), Salant & Rubinstein (2008 RES), Apesteguia & Ballester (2009 ET), Dean, Kibris, & Masatlioglu (2017 JET), Lim (2024 WP).